*International Journal of Theoretical Physics, Vol. 28, No. 9, 1989*

# From Unstable Minkowski Space to Inflation

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*Received June 20, 1989*

We present a perturbative approach to the equations describing the behavior of a quantum scalar field in a self-consistently generated Robertson-Walker universe. This approach throws new light on the significance of the Minkowskian instability and on the subtraction procedure which shows that a inflation cosmology is a possible future of the Minkowski space.

Traditional cosmology, as regulated by Einstein's equations, has its roots in a singular primordial event: the big bang. This appears to be inescapable, provided some general conditions on the matter energy-momentum tensor are fulfilled: the positivity Hawking–Penrose (HP) conditions, which correspond to rather "traditional" equations of state, usually encountered in the description of classical fluids. This situation is drastically altered in the framework of semiclassical gravity (Brout *et al.*, 1978, 1979*a,b*, 1980), wherein the matter sources are treated quantum mechanically. It appears indeed that the particle production mechanism which shows up in this context may lead to an effective equation of state (negative pressure associated with particle production) (Prigogine, 1947; Prigogine and Geheniau, 1986; Geheniau and Prigogine, 1986; Gunzig and Nardone, 1982, 1984; Gunzig *et al.*, 1987; Biran *et al.*, 1983), which violates the premises of the HP singularity theorem.

Let us analyze more closely how particle production together with its feedback reaction on space-time expansion, responsible for this creation, may lead to the avoidance of the initial singularity.

In order to avoid nonessential technical difficulties, the considerations will be restricted to the case of a single scalar matter field imbedded in a homogeneous and isotropic universe. The matter-gravitational action then

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0020-7748/89/0900-0935$06.00/0 

takes the form

$$S=\frac{-1}{2\kappa}\int d^4x\,\sqrt{-g}\,R+\tfrac{1}{2}\int d^4x\,\sqrt{-g}\left(\Psi_\mu\Psi_\nu g^{\mu\nu}-m^2\Psi^2+\frac{R}{6}\Psi^2\right)\quad(1)$$

and the metric, which is conformally flat, is given by

$$ds^2=d\tau^2-a(t)^2\,dl^2=l^2\Phi(t)^2(dt^2-dl^2)=l^2\Phi(t)^2\tilde{g}_{\alpha\beta}\,dx^\alpha\,dx^\beta\qquad(2)$$

The tilde refers to Minkowskian quantities. The conformal factor $l\Phi$ permits us to rescale the matter field $\Psi$ in the following way:

$$\psi=l\Phi\Psi,\qquad 6l^2=\kappa\qquad(3)$$

The action reduces then to the simple form

$$S=\tfrac{1}{2}\int d^4x\,\sqrt{-g}\left(\psi_\alpha\psi_\beta\tilde{g}^{\alpha\beta}+\frac{\tilde{R}}{6}\psi^2-m^2l^2\Phi^2\psi-\Phi_\alpha\Phi_\beta\tilde{g}^{\alpha\beta}-\frac{\tilde{R}}{6}\Phi^2\right)\quad(4)$$

The corresponding equations of motion are, for $\psi$,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\psi-\frac{\tilde{R}}{6}\psi+m^2l^2\psi\Phi^2=0\qquad(5)$$

and the related Einstein equations are

$$T_{\mu\nu}(\Phi)=T_{\mu\nu}(\psi)+\tfrac{1}{2}g_{\mu\nu}m^2l^2\psi^2\Phi^2\qquad(6)$$

where

$$T_{\mu\nu}(\Phi)=\Phi_\mu\Phi_\nu-\tfrac{1}{6}\Phi^2_{;\mu;\nu}+\tfrac{1}{6}g_{\mu\nu}D\Phi^2-\tfrac{1}{2}g_{\mu\nu}\Phi^\alpha\Phi_\alpha+\tfrac{1}{6}(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R)\Phi^2\quad(7)$$

The traditional covariant conservation law $T^\nu_{\mu;\nu}=0$ is now replaced by its Minkowskian version:

$$\partial_\nu(T^\nu_\mu(\Phi)-T^\nu_\mu(\psi))=0\qquad(8)$$

Hence, from the restricted point of view of the energy-momentum conservation laws, nothing forbids *a priori* a nontrivial realization of these laws. In such a case, the total $T^\nu_\mu$ would permanently keep its Minkowskian zero energy-momentum value. In spite of this global energy-momentum degeneracy of the matter-gravitational system with respect to empty Minkowskian space, the two interacting parts $T^\psi_{\mu\nu}$ and $T^\Phi_{\mu\nu}$ will acquire nonzero values starting at a given time, call it $t_0$. Hence, for $t\le t_0$ the system is strictly Minkowskian, with both parts of its total energy-momentum vanishing separately, whereas for $t\ge t_0$, the two matter and gravitational contributions $T^\psi_{\mu\nu}$ and $T^\Phi_{\mu\nu}$, respectively, are separately nonvanishing, although their sum is still zero. This possible realization of the conservation laws would then correspond to a simultaneous emergence of massive matter constituents together with a curved space-time background out of an empty, flat Minkowski space-time vacuum.

The semiclassical Einstein equations are

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\kappa\langle T_{\mu\nu}\rangle_s \tag{9}$$

where the expectation value for the energy-momentum tensor $\langle T_{\mu\nu}\rangle$ must be defined by a suitable subtraction procedure. Homogeneity and isotropy of the metric lead to a quite simple quantification procedure for the matter field $\psi$. Let us define

$$\psi(xt)^*\int\frac{d^3p}{(2\pi)^{3/2}}[\exp(i\mathbf{px})]\phi_p(t) \tag{10}$$

The equation of motion on $\psi$ leads then to an equation on $\phi_p(t)$:

$$\partial_t^2\phi_p(t)+[p^2+m^2a^2(t)]\phi_p=0 \tag{11}$$

Let us define time-dependent operators $A(t)$ and $A^\dagger(t)$,

$$\phi_p=(2\omega)^{-1/2}[A^\dagger_{-p}(t)+A_p(t)] \tag{12}$$

$$\partial_t\phi_p=i(\omega/2)^{1/2}[A^\dagger_{-p}(t)-A_p(t)] \tag{13}$$

where $\omega^2(t)=p^2+m^2a^2(t)$.

The operators $A$, $A^\dagger$ diagonalize the Hamiltonian

$$H=\int\frac{d^3p}{(2\pi)^{3/2}}\,\omega_p(t)(A_p^\dagger A_p+A_pA_p^\dagger) \tag{14}$$

and equation (12) becomes

$$\partial_tA^\dagger_{-p}=i\omega A^\dagger_{-p}+\frac{\dot\omega}{2\omega}A_p \tag{15}$$

Equation (15) shows directly pair creation of what we can call instantaneous particles,

$$\partial_t(A_p^\dagger A_p)=\frac{\dot\omega}{2\omega}(A_pA_{-p}+A_p^\dagger A^\dagger_{-p}) \tag{16}$$

Let us take the vacuum state $|\Omega\rangle$ as the state of the matter defined by

$$A(0)|\Omega\rangle=0 \tag{17}$$

We can then compute the expectation value of $T_{\mu\nu}$ on this state, and derive the corresponding "phenomenological" energy density and pressure by

$$\phi_p(t)=A^\dagger_{-p}(0)\xi_p(t)+A_p(0)\xi_p^*(t) \tag{18}$$

$$\dot\phi_p(t)=A^\dagger_{-p}(0)\dot\xi_p(t)+A_p(0)\dot\xi^*(t) \tag{19}$$

and

$$\rho=\frac{1}{2a^4(t)}\left\{\int_0^\infty \frac{p^2\,dp}{2\pi^2}[|\dot{\xi}|^2+\omega^2(t)|\xi|^2]\right\}_s \tag{20a}$$

$$p=\frac{1}{6a^4(t)}\left\{\int_0^\infty \frac{p^2\,dp}{2\pi^2}(|\dot{\xi}|^2+[p^2-m^2a^2(t)]|\xi|^2)\right\}_s \tag{20b}$$

$$\rho-3p=\frac{1}{2a^4}\left\{\int_0^\infty \frac{p^2\,dp}{2\pi^2}2m^2a^2(t)|\xi|^2\right\}_s \tag{20c}$$

where

$$\partial_t\xi+\omega^2(t)\xi=0,\qquad \xi^*\partial_t\xi-cc=i \tag{21}$$

The only relevant Einstein equation is then

$$3\frac{\dot{a}^2}{a^4}=\kappa\rho=\frac{\kappa}{2a^4(t)}\left\{\int_0^\infty \frac{p^2\,dp}{2\pi^2}[|\dot{\xi}|^2+\omega^2(t)|\xi|^2]\right\}_s \tag{22}$$

and the conservation law leads to

$$\partial_t(\rho a^4)=\dot{a}a^3(\rho-3p) \tag{23}$$

At this stage their are two obvious problems to be solved: to exhibit solutions of equation (22) and to select a subtraction procedure leading to finite expression for $\rho$ and $p$.

If we require that Minkowski space-time $[a(t)=1]$ must be a (trivial) solution, we need to subtract the vacuum fluctuation contribution to $\rho$ and $p$ in such a way as to obtain $\rho=p=0$,

$$\begin{aligned}\rho_{\text{Mink}}&=\frac{1}{2}\left\{\int_0^\infty \frac{p^2\,dp}{2\pi^2}\frac{\omega_0}{2}\right\}_s=0\\ (\rho-3p)_{\text{Mink}}&=m^2\left\{\int_0^\infty \frac{p^2\,dp}{2\pi^2}\frac{1}{2\omega}\right\}_s=0\end{aligned} \tag{24}$$

Let us analyze a small perturbation around Minkowski space,

$$a(t)=1+\delta(t) \tag{25}$$

Using the traditional perturbation theory for the expectation value (but not for the $S$-matrix), we obtain for $\langle\psi^2\rangle$

$$\langle\Omega|\psi^2|\Omega\rangle=\langle\Omega|\left\{\psi_i^2+i\int_0^t dt_1\,[V_i(t_1);\psi_i^2(x,t)]+\cdots\right\}|\Omega\rangle \tag{26}$$

where the subscript "$i$" indicates interaction picture. We have

$$V=\tfrac{1}{2}\int d^3x\,m^2\Delta(t)\psi^2$$

where

$$(1+\delta)^2 = 1+\Delta(t)$$

It follows, order by order,

$$\begin{aligned}\langle\psi^2\rangle_0 &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\\ \langle\psi^2\rangle_1 &= -m^2\int_0^t dt_1\,\Delta(t_1)\int\frac{d^3p}{(2\pi)^3}\frac{\sin 2\omega_p(t-t_1)}{2\omega_p}\end{aligned} \tag{27}$$

Integration by parts leads to

$$\langle\psi^2\rangle_1 = -m^2\Delta(t)\int\frac{d^3p}{(2\pi)^3}\frac{1}{4\omega^3}+\text{finite term} \tag{28}$$

The first term of order 1 is divergent, but it can be easily seen that it is the first-order perturbation of the "instantaneous" vacuum fluctuation:

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p(t)} \tag{29}$$

This prompts us to define a "minimal" subtraction (which can be proven to be consistent to all orders) by the requirement

$$\langle\Omega|O(t)|\Omega\rangle_s = \langle\Omega|O(t)|\Omega\rangle - \langle\Omega_t|O(t)|\Omega_t\rangle \tag{30}$$

$|\Omega_t\rangle$ is the "instantaneous vacuum" state. This subtraction is equivalent to including an infinite cosmological counterterm in the $n$-dimensional action

$$S = S_n(\psi) - S_n(\text{grav}) - \tfrac{1}{2}C_\infty\int d^nx\sqrt{-g}$$

with

$$C_\infty = -\frac{m^n}{2V_n}\frac{\Gamma((2-n)/2)\Gamma((n-1)/2)}{\Gamma(1/2)} \tag{31}$$

Using this subtraction procedure, we can linearize the Einstein equation

$$6\partial_t^2\delta = \kappa m^2\frac{m^2}{4\pi^2}\int_0^\infty\frac{p^2\,dp}{2\omega_p^3}\int_0^t dt_1\,\dot{\delta}(t_1)\cos 2\omega_p(t-t_1) \tag{32}$$

The Laplace transformation

$$\delta(s) = \int_0^\infty e^{-st}\ddot{\delta}(t)\,dt \tag{33}$$

then reduces equation (32) to the simple algebraic equation

$$\delta(s)=\frac{f(s)}{1-g(s)} \tag{34}$$

where

$$g(s)=\frac{\kappa m^4}{48\pi^2}\frac{2}{s^3}\left[(4m^2+s^2)^{1/2}\ln\left(\frac{s}{2m}+\sqrt{\frac{s^2}{4m^2}+1}\right)-s\right] \tag{35}$$

The existence of a real zero in the denominator (Gunzig and Nardone, 1982, 1984; Biran *et al.*, 1983) in (34) will then exhibit the instability of Minkowski space. Because $g(s)$ is a decreasing function in the domain $s\colon 0\to\infty$, the mentioned instability reduces simply to

$$1\le g_{\max}=\frac{\kappa m^2}{288\pi^2} \tag{36}$$

So the instability of Minkowski space shows up as soon as $\kappa m^2\ge 288\pi^2$. This instability corresponds to a global fluctuation $\delta(t)$, but we expect that for inhomogeneous fluctuation $\delta(x, t)$, this will not change qualitatively.

It was previously proven that, using the same subtraction procedure, the Euclidean de Sitter space, namely $a(t)=t_0t^{-1}$, is an exact self-consistent solution if $\kappa m^2>288\pi^2$. These two results are consistent. If the mass is below the threshold mass $(288\pi^2\kappa^{-1})^{1/2}$, Minkowski is the only solution; on the contrary, when the mass is above this threshold, then the universe transits to an "inflationary" stage: the de Sitter universe.

We have also developed a WKB-like analysis for the equation (21) in order to exhibit the solution $\xi(t)$ for any function $a(t)$. This leads to

$$d_\tau(2\nu f_\tau^2+\tfrac{1}{2}\mu f^2+\tfrac{1}{6}\nu f^6)=-12\nu f^2 f_\tau^2 \tag{37}$$

where

$$\nu=\frac{\kappa}{2880\pi^2},\qquad \mu=1-\frac{\kappa m^2}{288\pi^2},\qquad f^2=H=a^{-1}\frac{da}{d\tau}$$

and $\tau$ is the cosmological time. Equation (37) shows directly that $\mu<0$ corresponds to a symmetry-breaking mechanism: $f=0$ (Minkowski space) is no longer the stable minimum and the universe expands until it reaches the stable de Sitter regime (Spindel, 1981) with $H=(-\mu/\nu)^{1/2}$.

## REFERENCES

Biran, B., Brout, R., and Gunzig, E. (1983). *Physics Letters B*, **125**, 399–402.
Brout, R., Englert, F., and Gunzig, E. (1978). *Annals of Physics*, **115**, 78–106.
Brout, R., Englert, F., and Gunzig, E. (1979*a*). *General Relativity and Gravitation*, **1**, 1–5.

Brout, R., Englert, F., and Spindel, P. (1979*b*). *Physical Review Letters*, **43**, 417.

Brout, R., *et al.* (1980). *Nuclear Physics B*, **170**, 228-264.

Geheniau, J., and Prigogine, I. (1986). *Foundations of Physics*, **16**, 437-445.

Gunzig, E., and Nardone, P. (1982). *Physics Letters B*, **118**, 324-326.

Gunzig, E., and Nardone, P. (1984). *General Relativity and Gravitation*, **16**, 305-309.

Gunzig, E., Geheniau, J., and Prigogine, I. (1987). *Nature*, **330**, 621-624.

Prigogine, I. (1947). *Etude Thermodynamique des Phénomènes Irréversibles*, Desoer, Liège, Belgium.

Prigogine, I., and Geheniau, J. (1986). *Proceedings of the National Academy of Sciences of the USA*, **83**, 6245-6249.

Spindel, P. (1981). *Physics Letters*, **107**, 361-363.